$\mathbb{R}^3$: 0 1 2 3

$$Fn \xrightarrow{\text{grad}} VF \xrightarrow{\text{curl}} VF \xrightarrow{\text{div}} Fn$$

① $\text{curl}(\text{grad}(f)) = 0$

$\text{div}(\text{curl}(\vec{F})) = 0$

② $f(\gamma(b)) - f(\gamma(a)) = \int_\gamma \text{grad}(f)$

$$\int_{\partial R} \vec{F} = \iint_R \text{curl}(\vec{F})$$

$$\iint_{\partial D} \vec{F} = \iiint_D \text{div}(\vec{F})$$

Q1 $\nabla\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H})$

Ans: $\text{L.H.S} = \begin{vmatrix} \partial_x & \partial_y & \partial_z \\ E_1 & E_2 & E_3 \\ H_1 & H_2 & H_3 \end{vmatrix} = \partial_x(E_2H_3 - E_3H_2) - \partial_y(E_1H_3 - E_3H_1) + \partial_z(E_1H_2 - E_2H_1)$

$$\text{R.H.S.} = \begin{vmatrix} H_1 & H_2 & H_3 \\ \partial_x & \partial_y & \partial_z \\ E_1 & E_2 & E_3 \end{vmatrix} - \begin{vmatrix} E_1 & E_2 & E_3 \\ \partial_x & \partial_y & \partial_z \\ H_1 & H_2 & H_3 \end{vmatrix}$$

$$\begin{vmatrix} H_1 & H_2 & H_3 \\ \partial_x & \partial_y & \partial_z \\ E_1 & E_2 & E_3 \end{vmatrix} = H_1(\partial_y E_3 - \partial_z E_2) - H_2(\partial_x E_3 - \partial_z E_1) + H_3(\partial_x E_2 - \partial_y E_1)$$

$$\begin{vmatrix} E_1 & E_2 & E_3 \\ \partial_x & \partial_y & \partial_z \\ H_1 & H_2 & H_3 \end{vmatrix} = E_1(\partial_y H_3 - \partial_z H_2) - E_2(\partial_x H_3 - \partial_z H_1) + E_3(\partial_x H_2 - \partial_y H_1)$$

$\Rightarrow$ Just compare

☆ Remember not expanding along the 2nd or the 3rd row.

Q2 $\oint f \nabla g \, d\vec{s} = \iint_S \nabla f \times \nabla g \cdot d\vec{\sigma}$

Ans: We need to show that $\nabla \times (f \nabla g) = \nabla f \times \nabla g$

$$\nabla \times (f \nabla g) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f\partial_x g & f\partial_y g & f\partial_z g \end{vmatrix}$$

$$= \vec{i}\left(\partial_y (f\partial_z g) - \partial_z (f\partial_y g)\right) - \vec{j}\left(\partial_x (f\partial_z g) - \partial_z (f\partial_x g)\right)$$
$$+ \vec{k}\left(\partial_x (f\partial_y g) - \partial_y (f\partial_x g)\right)$$
$$= \vec{i}\left(\partial_y f \cdot \partial_z g - \partial_z f \, \partial_y g\right) - \vec{j}\left(\partial_x f \cdot \partial_z g - \partial_z f \cdot \partial_x f\right)$$
$$+ \vec{k}\left(\partial_x f \cdot \partial_y g - \partial_y f \cdot \partial_x g\right)$$

And $$\nabla f \times \nabla g = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x f & \partial_y f & \partial_z f \\ \partial_x g & \partial_y g & \partial_z g \end{vmatrix}$$

$$= \vec{i}\left(\partial_y f \cdot \partial_z g - \partial_z f \, \partial_y g\right) - \vec{j}\left(\partial_x f \cdot \partial_z g - \partial_z f \cdot \partial_x f\right)$$
$$+ \vec{k}\left(\partial_x f \cdot \partial_y g - \partial_y f \cdot \partial_x g\right)$$

Q3. Divergence in Cylindrical coordinates.

Let $\vec{r} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$, $\vec{\theta} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$

Suppose $\vec{F}$ is a vector field given by

$$\vec{F}(r,\theta) = F_r\vec{r} + F_\theta\vec{\theta} + F_z\cdot\vec{k}$$

Show that $\nabla\cdot\vec{F} = \frac{1}{r}\cdot\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}$

Ans: Write $\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$.

Then $F_x = F_r\cos\theta - F_\theta\sin\theta$

$F_y = F_r\sin\theta + F_\theta\cos\theta$

We first compute $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}$:

$$r^2 = x^2+y^2 \Rightarrow \begin{cases} 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \\ 2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta \end{cases}$$

$$x\tan\theta = y \Rightarrow \begin{cases} \tan\theta + x\sec^2\theta\frac{\partial\theta}{\partial x} = 0 \Rightarrow \frac{\partial\theta}{\partial x} = -\frac{\tan\theta}{x\sec^2\theta} = -\frac{\sin\theta}{r} \\ x\sec^2\theta\cdot\frac{\partial\theta}{\partial y} = 1 \Rightarrow \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r} \end{cases}$$

Now, $\nabla\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

$$\frac{\partial F_x}{\partial x} = \frac{\partial F_x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial F_x}{\partial\theta}\frac{\partial\theta}{\partial x}$$

$$= \left(\frac{\partial F_r}{\partial r}\cos\theta - \frac{\partial F_\theta}{\partial r}\sin\theta\right)\cos\theta + \left(\frac{\partial F_r}{\partial\theta}\cos\theta - F_r\sin\theta - \frac{\partial F_\theta}{\partial\theta}\sin\theta - F_\theta\cos\theta\right)\left(\frac{-\sin\theta}{r}\right)$$

$$\frac{\partial F_y}{\partial y} = \frac{\partial F_y}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial F_y}{\partial\theta}\cdot\frac{\partial\theta}{\partial y}$$

$$= \left(\frac{\partial F_r}{\partial r}\sin\theta + \frac{\partial F_\theta}{\partial r}\cos\theta\right)\sin\theta + \left(\frac{\partial F_r}{\partial\theta}\sin\theta + F_r\cos\theta + \frac{\partial F_\theta}{\partial\theta}\cos\theta - F_\theta\sin\theta\right)\left(\frac{\cos\theta}{r}\right)$$

Therefore, $\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \frac{\partial F_r}{\partial r} + \frac{1}{r}F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta}$

$$= \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta}$$

Q4. Let $\vec{F}$ be a vector field on $\mathbb{R}^3 - \{0\}$ defined by

$$\vec{F}(\vec{v}) = \frac{1}{\|\vec{v}\|^3}\vec{v}$$

Let D be a region that contains the origin in its interior, $S = \partial D$

Show that $\iint_S \vec{F} = 4\pi$

Ans: Case that D is a ball centered at the origin with radius ε.

parametrization of S:

$\gamma(\theta, \varphi) = \varepsilon(\cos\varphi\cos\theta, \cos\varphi\sin\theta, \sin\varphi)$

$\gamma_\theta = \varepsilon(-\cos\varphi\sin\theta, \cos\varphi\cos\theta, 0)$

$\gamma_\varphi = \varepsilon(-\sin\varphi\cos\theta, -\sin\varphi\sin\theta, \cos\varphi)$

$\vec{F} = \frac{1}{\varepsilon^2}(\cos\varphi\cos\theta, \cos\varphi\sin\theta, \cos\varphi)$

$$\Rightarrow \vec{F}\cdot(\gamma_\theta \times \gamma_\varphi) = \begin{vmatrix} \cos\varphi\cos\theta & \cos\varphi\sin\theta & \sin\varphi \\ -\cos\varphi\sin\theta & \cos\varphi\cos\theta & 0 \\ -\sin\varphi\cos\theta & -\sin\varphi\sin\theta & \cos\varphi \end{vmatrix}$$

$$= \sin\varphi(\sin\varphi\cos\varphi) + \cos\varphi(\cos^2\varphi)$$

$$= \cos\varphi$$

$$\Rightarrow \iint_S \vec{F} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi} \cos\varphi \, d\theta \, d\varphi$$

$$= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \, d\varphi$$

$$= 4\pi$$

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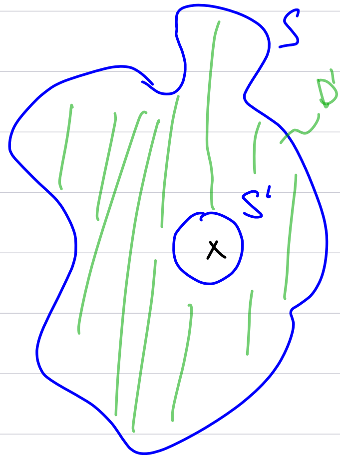

Ans: General case

$$\iint_{S'} \vec{F} = 4\pi$$

Stokes' thm

$$\iint_S \vec{F} = \iint_{S'} \vec{F} + \iiint_{D'} \operatorname{div}(\vec{F})$$

Now we compute $\operatorname{div}(\vec{F})$

$$\vec{F} = \left(\underbrace{\frac{x}{(x^2+y^2+z^2)^{3/2}}}_{F_1}, \underbrace{\frac{y}{(x^2+y^2+z^2)^{3/2}}}_{F_2}, \underbrace{\frac{z}{(x^2+y^2+z^2)^{3/2}}}_{F_3} \right)$$

$$\frac{\partial F_1}{\partial x} = \frac{1}{(x^2+y^2+z^2)^{3/2}} + x\,\frac{(-3/2)}{(x^2+y^2+z^2)^{5/2}}(2x)$$

$$= \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\Rightarrow \operatorname{div}(\vec{F}) = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} = \frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$$